Discrete Mathematics 122 (1993) 113–136
North-Holland

An interval digraph in relation to its associated bipartite graph

S. Das and M. Sen

Department of Mathematics, North Bengal University, Raja Rammohunpur, Darjeeling, West Bengal, India - 734430

Received 3 February 1989
Revised 21 June 1991

Abstract

The *intersection digraph* of a family of ordered pairs of sets $\{(S_v, T_v): v \in V\}$ is the digraph $D(V, E)$ such that $uv \in E$ if and only if $S_u \cap T_v \neq \emptyset$. *Interval digraphs* are those intersection digraphs for which the subsets are intervals on the real line. In a previous paper, they were characterized in terms of Ferrers digraphs and a close relationship was obtained between an interval digraph and a digraph of Ferrers dimension 2.

In order to characterize a digraph D of Ferrers dimension 2, Cogis associated an undirected graph $H(D)$ with D in a suitable way, the vertices of $H(D)$ corresponding to the zeros of the adjacency matrix of D. He proved that D has Ferrers dimension at most 2 if and only if $H(D)$ is biparite.

Depending on the above characterization, this paper first obtains some properties of a digraph of Ferrers dimension 2; then it is shown how the notion of interior edges is related to an interval digraph.

1. Introduction

Previously [1, 11], the idea of intersection digraphs analogous to the well-known concept of intersection graphs was introduced. Given a family $\mathscr{F}$ of ordered pairs of subsets $\{(S_v, T_v): v \in V\}$, $S_v, T_v \subset X$, the *intersection digraph* of the family $\mathscr{F}$ is the digraph $D(V, E)$ with vertex set V and the edge set E defined by $uv \in E$ if and only if S_u has a non-empty intersection with T_v. The digraphs are finite, have no multiple edges but loops are permitted. Beineke and Zamfirescu [1] introduced and used this concept while characterizing line digraphs. *Interval digraphs* are those intersection digraphs for which the subsets are intervals on the real line. Several characterizations of interval digraphs were obtained in [11, 12]. In [11], they were characterized in terms of Ferrers digraphs.

Correspondence to: S. Das, Department of Mathematics, North Bengal University, Raja Rammohunpur, Darjeeling, West Bengal, India-734 430.

Recall that a digraph $D(V, E)$ is a Ferrers digraph if $ab \in E$ and $cd \in E \Rightarrow ad \in E$ or $cb \in E$, for all $a, b, c, d \in V$. (Note that a or c may be equal to either b or d). Several authors dealt with this concept of Ferrers digraphs in varied contexts and [2, 3, 6, 7, 9, 10] may be seen for reference. It was Riguet [10] who introduced Ferrers digraphs and characterized these digraphs as those in which the successor sets (or equivalently the predecessor sets) are linearly ordered by inclusion. Equivalently the rows and columns of the adjacency matrix can be (independently) permuted in such a way that every 1 has all positions 1 below and to the left of it, that is, the ones are clustered in the lower left (alternatively, the rows and columns of the adjacency matrix can be (independently) permuted in such a way that every 1 has all positions 1 above and to the right of it, that is, the ones are clustered in the upper right). Any digraph D is the intersection of a (finite) number of Ferrers digraphs [2, 4] and the minimum cardinality of such Ferrers digraphs is the Ferrers dimension (F.D.) of D. The digraphs with F.D.2 were characterized independently by Cogis [5] and also by Doignon, Ducamp and Falmagne [6] in different contexts. For additional references see Golumbic [8] and West [13].

Ferrers digraphs were also characterized immediately from its definition by Riguet [10] in terms of a forbidden submatrix of its adjacency matrix. In this paper, we shall frequently use the adjacency matrices of a digraph D and its complement $\bar{D}$ and shall adopt the convention to use the same matrix $A = (a_{ij})$ where

$$a_{ij} = \begin{cases} D, & \text{where } v_i v_j \in E, \\ \bar{D}, & \text{where } v_i v_j = \bar{E} \end{cases}$$

representing either of the digraphs D and its complement $\bar{D}$. Note that the variable D and $\bar{D}$ have the values 1 and 0 respectively for the digraph D, while they have the values 0 and 1 respectively for the digraph $\bar{D}$. A 2×2 permutation matrix i.e., a submatrix of the form

$$\begin{pmatrix} D & \bar{D} \\ \bar{D} & D \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \bar{D} & D \\ D & \bar{D} \end{pmatrix}$$

is called an *obstruction* in the matrix. Alternatively, two edges xy and zt of D are said to form an obstruction, written $(xy * zt)_D$ or $xy * zt$, if none of xt and zy belong to D. i.e. $xy \in D$, $zt \in D$ but $xt \in \bar{D}$ and $zy \in \bar{D}$. Cogis calls them *F-incompatible*. A digraph D is a Ferrers digraph if and only if A has no obstruction, in other words, if and only if the adjacency matrix has no 2×2 permutation matrix. It follows immediately that a digraph D is a Ferrers digraph if and only if its complement is also so. In order to characterize a digraph of F.D.2, Cogis [5] defined an undirected graph $H(D)$, *the graph associated with* D whose vertices correspond to the $\bar{D}$'s of A, with two such vertices ($\bar{D}$'s) joined by an edge if the corresponding $\bar{D}$'s belong to an obstruction. Cogis [5] proved that a finite digraph D has Ferrers dimension 2 at most iff $H(D)$ is bipartite. Then he used this result to obtain a recognition algorithm for a digraph of F.D.2 in a polynomial time. The same characterization was obtaind in a more general form by Doignon, Ducamp and Falmagne [6] where the set is not restricted to be finite. The graph $H(D)$ may have more than one (connected) component; besides it may have one or more isolated vertices (corresponding to the $\bar{D}$'s of A which do not belong to any obstruction). The graph obtained by deleting the isolated vertices from $H(D)$ is denoted by $H_b(D)$. It is called the *bare graph associated with D* (Doignon et al. [6]).

It was proved in [11] that a digraph of F.D.$\leqslant 2$ is equivalent to the existence of independent row and column permutations of the adjacency matrix so that the resulting matrix has no $\bar{D}$ with a D below it and another D to its right. In other words, corresponding to any $\bar{D}$ in the rearranged matrix, either every entry below it is $\bar{D}$ or every entry to its right is $\bar{D}$ ('or' being inclusive). We shall refer to this property as *F_2-property* for the rearranged adjacency matrix of the digraph and the rearranged matrix as an *F_2-matrix* of the digraph. It is to be noted in this connection that in an F_2-matrix, a pair of $\bar{D}$'s forming an obstruction must have the form

$$\begin{pmatrix} D & \bar{D} \\ \bar{D} & D \end{pmatrix}$$

because the presence of the other form

$$\begin{pmatrix} \bar{D} & D \\ D & \bar{D} \end{pmatrix}$$

violates its F_2-peoperty. It was also shown in [11] that an interval digraph is necessarily a digraph of F.D. at most 2; but the converse is not true. As a matter of fact, it was proved that a digraph D is an interval digraph if and only if it is the intersection of two Ferrers digraphs whose union is complete or, equivalently, if and only if its complement $\bar{D}$ is the union of two disjoint Ferrers digraphs (since the complement of a Ferrers digraph is a Ferrers digraph).

Given a digraph D of F.D.2 and a realization of $\bar{D}$ as the union of two Ferrers digraphs, we first introduce in this paper the notion of interior edges of these two Ferrers digraphs (with reference to the given realization). We use this concept to obtain some properties of a digraph of F.D.2 and then we show how the notion of interior edges is related to an interval digraph.

2. Interior edges

We begin with the following well-known theorem.

Theorem 1 (Cogis, Doignon et al.). *A digraph D is of Ferrers dimension at most 2 iff $H(D)$ is bipartite.*

Let $D(V, E)$ be a digraph of F.D.2 so that $H(D)$ is a bipartite graph. We shall denote the set of all isolated vertices of $H(D)$ by $\boldsymbol{I}(H)$ or by $\boldsymbol{I}$, and a bicolouration of $H_b(D)$ by

$(\boldsymbol{R}, \boldsymbol{C})$. Recall that a colouration of a graph is an assignment of colours to its vertices so that no two adjacent points have the same colour. Naturally, a bicolourable graph uses two colours only. If $H_b(D)$ has more than one connected component $H_1, \ldots, H_p$, a biocolouration of H_i will be denoted by $(\boldsymbol{R}_i, \boldsymbol{C}_i)$. It is evident that $\boldsymbol{R} = \bigcup_1^p \boldsymbol{R}_i$ and $\boldsymbol{C} = \bigcup_1^p \boldsymbol{C}_i$ for some labelling of the bicolouration $(\boldsymbol{R}_i, \boldsymbol{C}_i)$ of H_i. We shall also denote the elements of the sets $\boldsymbol{R}, \boldsymbol{C}, \boldsymbol{R}_i, \boldsymbol{C}_i$ or $\boldsymbol{I}$ by the corresponding capital letters R, C, R_i, C_i or I respectively. The stable sets $\boldsymbol{R}_i$ and $\boldsymbol{C}_i$ will be called the fragments of $H(D)$ (Cogis calls them p-colours). The two fragments $\boldsymbol{R}_i$ and $\boldsymbol{C}_i$ (for the same i) will be called conjugate to each other. For a digraph $D(V, E)$ of F.D.2, $\bar{D} = G_1(V, E_1) \cup G_2(V, E_2)$, where G_1 and G_2 are two Ferrers digraphs. Since G_k's $(k = 1, 2)$ are Ferrers digraphs. any two edges of G_k cannot form an obstruction in $A(G_k)$ and so in $A(G)$. Since again G_1 and G_2 are subdigraphs of $\bar{D}$, two edges ab and cd forming an obstruction in $\bar{D}$ must not belong to the same G_k $(k = 1, 2)$, i.e., $ab \in E_1$ (or E_2) $\Rightarrow cd \in E_2$ (or E_1). Thus if $H_b(D)$ has more than one component H_i $(i = 1, \ldots, p)$, then given $G_1(V, E_i)$ and $G_2(V, E_2)$ whose union is $\bar{D}$ there exist some labelling $(\boldsymbol{R}_i, \boldsymbol{C}_i)$ of the bicolouration of H_i such that $\boldsymbol{R} = \bigcup_1^p \boldsymbol{R}_i \subset E_i$ and $\boldsymbol{C} = \bigcup_1^p \boldsymbol{C}_1 \subset E_2$. So if we want to cover $\bar{D}$ by two Ferrers digraphs, we should consider the fragments $(\boldsymbol{R}_i, \boldsymbol{C}_i)$ of $H(D)$ (which, in turn, yields a bicolouration of $H_b(D)$). On the other hand, however, any bicolouration of $H_b(D)$ does not necessarily lead to a covering of $\bar{D}$ by two Ferrers digraphs. It is easily verified by considering the simple digraph

	v_1	v_2	v_3
v_1	D	$\bar{D}$	$\bar{D}$
v_2	$\bar{D}$	D	$\bar{D}$
v_3	$\bar{D}$	$\bar{D}$	D .

For the digraph D, $H(D)$ is the graph consisting of three disconnected edges $\{v_1v_2, v_2v_1\}, \{v_1v_3, v_3v_1\}$ and $\{v_2v_3, v_3v_2\}$.

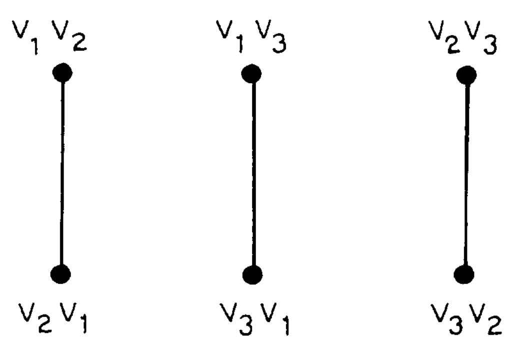


Fig. 1. The graph $H(D)$.

If we consider

$$R_1=\{v_1v_2\},\qquad R_2=\{v_3v_1\},\qquad R_3=\{v_2v_3\},$$
$$C_1=\{v_2v_1\},\qquad C_2=\{v_1v_3),\qquad C_9=\{v_3v_2\};$$

then the bicolouration $E_1=R_1\cup R_2\cup R_3$ and $E_2=C_1\cup C_2\cup C_3$ does not lead to a covering of $\bar{D}$ into two Ferrers digraphs whereas it does if we choose $E_1=R_1\cup C_2\cup R_3$ and $E_2=C_1\cup R_2\cup C_3$.

While proving Theorem 1, Cogis [5] adopted a constructive method to show that there always exists a suitable bicolouration of $H_b(D)$ that yields a realization of $\bar{D}$ as the union of two Ferrers digraphs. As a matter of fact, he obtained the particular bicolouration $(\boldsymbol{R}, \boldsymbol{C})$ of $H_b(D)$ in such a way that adjoining all the edges of $\boldsymbol{I}(H)$ to each of $\boldsymbol{R}$ and C yielded the required Ferrers digraphs realization G_1 and G_2 so that $\bar{D}=G_1\cup G_2$ where $G_1=\boldsymbol{R}\cup\boldsymbol{I}(H)$ and $G_2=\boldsymbol{C}\cup\boldsymbol{I}(H)$. As we shall often require this result in our assertions, we state this property in the form of a proposition.

Proposition 1 (Cogis [5]). *For a digraph D of F.D.2, there exists a bicolouration* $(\boldsymbol{R}, \boldsymbol{C})$ *of* $H_b(D)$ *such that* $\boldsymbol{R}\cup\boldsymbol{I}(H)$ *and* $\boldsymbol{C}\cup\boldsymbol{I}(H)$ *are Ferrers relations; these relations in turn, yield a realization of* $\bar{D}$ *as the union of two Ferrers digraphs* G_1 *and* G_2, $\bar{D}=G_1\cup G_2$, *where*

$$G_1\boldsymbol{R}\cup\boldsymbol{I}(H)\quad\text{and}\quad G_2=\boldsymbol{C}\cup\boldsymbol{I}(H).$$

Such a bicolouration $(\boldsymbol{R}, \boldsymbol{C})$ of $H_b(D)$ for which $G_1=\boldsymbol{R}\cup\boldsymbol{I}(H)$ and $G_2=\boldsymbol{C}\cup\boldsymbol{I}(H)$ are Ferrers digraphs, will in our paper be termed a *satisfactory bicolouration*. The above result was independently proved by Doignon et al. [7] in the more general case when the set of vertices is not necessarily finite. Indeed they also prove that in a certain restricted case any bicolouration of $H_b(D)$ very well serves the purpose.

In this paper, by a *configuration of an adjacency matrix A*, we shall mean a submatrix of A obtained by any (independent) permutation of rows and of colums. But by *a configuration of an* F_2*-matrix F*, we shall, for convenience, mean a submatrix of F up to (independent) permutation of rows and columns so long as the rearranged permuted matrix retains its F_2-matrix structure (with the same labelling of R_i and C_i).

While the recognition of a digraph of F.D.2 requires the realization of its complement as the union of two Ferrers digraphs G_1 and G_2, not necessarily disjoint, such that $\bar{D}=G_1\cup G_2$, the problem for an interval digraph recognition, however, is to cover its complement by two Ferrers digraphs which should necessarily be disjoint, $\bar{D}=H_1\cup H_2$, $H_1\cap H_2=\emptyset$. This is equivalent to adjoining every edge $I\in\boldsymbol{I}(H)$ into only one of the two digraphs $G_1(V, \boldsymbol{R})$ and $G_2(V, \boldsymbol{C})$ for some bicolouration $(\boldsymbol{R}, \boldsymbol{C})$ of $H_b(D)$ so that they become two disjoint Ferrers digraphs.

In the following, we prove some elementary properties of a digraph of F.D.2, which will be required in the sequel.

Proposition 2. *Let D be a digraph of F.D.2. Then a $\bar{D}$ of A is an isolated vertex of $H(D)$, iff there exists an F_2-matrix of A in which no D lies below or to the right of the corresponding $\bar{D}$ in the matrix.*

Proof. *Sufficiency.* Let the rearranged matrix satisfying the F_2-property be denoted by $F=(a_{ij})$. Let a position a_{ij} in F be $\bar{D}$ with all the elements to the right of a_{ij} in the ith row and all the elements below a_{ij} in the jth column being $\bar{D}$. Evidently, a_{ij} cannot have any obstruction with any $\bar{D}$ to the right of jth column or any $\bar{D}$ below ith row. Now consider a $\bar{D}$ above the ith row and to the left of the jth column. Then since the matrix satisfies F_2-property, the two $\bar{D}$'s cannot form an obstruction. Hence the $\bar{D}$ corresponding to a_{ij} is an isolated vertex in $H(D)$.

Necessity. Let D be a digraph of F.D.2 and let a $\bar{D}$ of A be an isolated vertex in $H(D)$. Call it I. Rearrange the adjacency matrix to an F_2-matrix B. Then either every element to the right of I is $\bar{D}$ or every element below I is $\bar{D}$. First we prove the proposition for the case when every element to the right of I is $\bar{D}$. We shall show that the row corresponding to I can be sufficiently shifted so as to satisfy the condition of the proposition. Let a position below I be D. Since I does not belong to any obstruction all the 2×2 submatrix with these I and D must not be of the form

$$\begin{pmatrix} D & I \\ \bar{D} & D \end{pmatrix}.$$

That is, it must be of one of the forms

$$\begin{pmatrix} D & I \\ D & D \end{pmatrix} \quad \begin{pmatrix} \bar{D} & I \\ \bar{D} & D \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \bar{D} & I \\ D & D \end{pmatrix}.$$

Since the matrix is an F_2-matrix we can easily see that the row corresponding to I may be shifted to a position when all elements below I are $\bar{D}$ (and the F_2-property is retained). The other case when every element below I is $\bar{D}$ may be proved similarly. $\square$

Recall from Theorem 5 [11] that a digraph D is of F.D. at most 2 iff A can be rearranged in the form of an F_2-matrix. Let $\bar{\boldsymbol{R}}$ be the set of $\bar{D}$'s having a D somewhere below them and $\bar{\boldsymbol{C}}$ be the set of $\bar{D}$'s having a D somewhere to its right. For any 2×2 submatrix forming an obstruction, the $\bar{D}$'s must be an $\bar{R}$ in the upper right and a $\bar{C}$ in the lower left and these are the only edges in the bipartite graph $H(D)$. Note that the $\bar{D}$'s which have no D to the right or below are isolated points in $H(D)$. These $\bar{R}$'s together with the above $\bar{D}$'s generating isolated points yield a Ferrers digraph $G_1(V,E_1)$ and the C's together with those $\bar{D}$'s yield another Ferrers digraph $G_2(V,E_2)$ so that $\bar{D}=G_1\cup G_2$. Again note that there may be an $\bar{R}$ which has no obstruction with a $\bar{C}$ and vice versa so that some $\bar{R}$'s and $\bar{C}$'s may again be isolated points in $H(D)$. If $H_b(D)$ has more than one component H_i, $H_b(D)=\bigcup H_i$, and if $(\boldsymbol{R},\boldsymbol{C})$ be the bicolouration of $H_b(D)$ in the above F_2-matrix, $(\boldsymbol{R}_i,\boldsymbol{C}_i)$ the bicolouration of H_i then

$$\boldsymbol{R}=\bigcup \boldsymbol{R}_i\subset \bar{\boldsymbol{R}}\subset E_1,$$

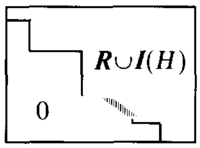
$$\boldsymbol{C}=\bigcup \boldsymbol{C}_i\subset \bar{\boldsymbol{C}}\subset E_2.$$

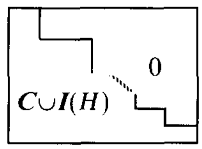
Again in the above F_2-matrix, any R_i has only R_j's and/or isolated vertices to its right and any C_i has only C_j's and/or isolated vertices below it; and any two members R_i and C_i belonging to two conjugate fragments such that (R, C) is an edge in $H(D)$ must appear as a submatrix

$$\begin{pmatrix} D & R_i \\ C_i & D \end{pmatrix}$$

in the above F_2-matrix (R_i appear in the upper right and C_i in the lower left).

Let $(\boldsymbol{R}, \boldsymbol{C})$ be a satisfactory bicolouration of $H_b(D)$ leading to a realization of $\bar{D} = G_1(V, E_1) \cup G_2(V, E_2)$ where $E_1 = \boldsymbol{R} \cup \boldsymbol{I}(H)$ and $E_2 = \boldsymbol{C} \cup \boldsymbol{I}(H)$. Let the rows and columns of $A(G_1)$ be so arranged that all the ones are clustered in the upper right. Similarly, the rows and columns of $A(G_2)$ are so arranged that all the ones are clustered in the lower left.

$A(G_1) =$ 

$A(G_2) =$ 

Definition 1. An edge corresponding to an $I \in G_1$ is said to be an *interior edge of* G_1, denoted I_r, if there exists a configuration of the form

$$\begin{pmatrix} R & I \\ 0 & R \end{pmatrix}$$

in $A(G_i)$; similarly, an $I \in G_2$ is said to be an interior edge of G_2, denoted I_c, if there exists a configuration of the form

$$\begin{pmatrix} C & 0 \\ I & C \end{pmatrix}$$

in $A(G_2)$. With reference to a particular realization of $\bar{D}$ as the union of G_1 and G_2, $\bar{D} = G_1 \cup G_2$, the set of all interior edges of G_1 will be called *interior of* G_1 and will be denoted by $\boldsymbol{I}_r(G_1)$ or $\boldsymbol{I}_r$ and all interior edges of G_2 will be called *interior of* G_2 and will be denoted by $\boldsymbol{I}_c(G_2)$ or $\boldsymbol{I}_c$. Note that the sets $\boldsymbol{I}_r$ and $\boldsymbol{I}_c$ are identified with reference to a particular realization of $\bar{D}$ and will change if the realization changes.

Proposition 3. *Let D be a digraph of F.D.2. Then for a satisfactory bicolouration of* $H_b(D)$ *the following two conditions are equivalent*:

(i) $I_r \cap I_c \neq \emptyset$ *and*

(ii) *A contains a configuration of the form*

$$\begin{bmatrix} D & D & R \\ D & D & C \\ C & R & I \end{bmatrix}$$

(*with respect to the same bicolouration*).

Proof. (ii)⇒(i) follows immediately from the definition of I_r and I_c.

(i)⇒(ii). Let $I \in I_r \cap I_c$. Then this I belongs to the configurations

$$\begin{bmatrix} R & I \\ 0 & R \end{bmatrix} \text{ and } \begin{bmatrix} C & 0 \\ I & C \end{bmatrix}$$

in $A(G_1)$ and $A(G_2)$ respectively. From Proposition 1, it follows that the adjacency matrix A contains both the configurations

$$\begin{bmatrix} R & I \\ D & R \end{bmatrix} \text{ and } \begin{bmatrix} C & D \\ I & C \end{bmatrix}$$

as well. Combining these two we can say that A has a configuration of the form

$$\begin{array}{c|ccc} & x' & y' & z' \\ \hline x & \cdot\cdot & C & D \\ y & R & I & C \\ z & D & R & \cdot\cdot \end{array}$$

where '$\cdot\cdot$' means we are yet to conclude anything about whether it is D or $\bar{D}$ (exclusively or inclusively). Let, if possible, the '$\cdot\cdot$' in the xx' position be $\bar{D}$. Then this $\bar{D}$ must be either R or C or I. If it is R or I, then the configuration

$$\begin{array}{c|cc} & x' & y' \\ \hline x & \cdot\cdot & C \\ z & D & R \end{array}$$

in A has a corresponding configuration in $A(G_1)$ given by

$$\begin{array}{c|cc} & x' & y' \\ \hline x & R/I & 0 \\ z & 0 & R \end{array}.$$

This yields an obstruction in $A(G_1)$ which is not possible. Similarly, if it is C or I, then the configuration

$$\begin{array}{c|cc} & x' & z' \\ \hline x & C/I & 0 \\ y & 0 & C \end{array}.$$

in $A(G_2)$ gives us an obstruction which is again not possible. So the '$\cdot\cdot$' in the xx' position must be D. Similarly, the '$\cdot\cdot$' in the zz' position must be D. Thus A has a configuration

$$\begin{array}{c|ccc} & x' & y' & z' \\ \hline x & D & C & D \\ y & R & I & C \\ z & D & R & D \end{array} \quad \text{or equivalently} \quad \begin{array}{c|ccc} & z' & x' & y' \\ \hline z & D & D & R \\ x & D & D & C \\ y & C & R & I \end{array} \qquad \square$$

Proposition 4. *Let D be a digraph of F.D.2. If $I_r \cap I_c \neq \emptyset$ for a certain satisfactory bicolouration $(\boldsymbol{R}, \boldsymbol{C})$ of $H_b(D)$ then the same is true for any other satisfactory bicolouration of $H_b(D)$.*

To prove this we require the following lemma.

Lemma 1. *Let D be a digraph of F.D.2 with a satisfactory bicolouration $(\boldsymbol{R}, \boldsymbol{C})$ of $H_b(D)$, $\boldsymbol{R} = \bigcup \boldsymbol{R}_i$, $\boldsymbol{C} = \bigcup \boldsymbol{C}_i$ and the rearranged F_2-matrix F have a configuration of any of the forms*

$$\text{(i)} \quad \begin{array}{c|ccc} & x' & y' & z' \\ \hline x & D & D & X \\ y & D & D & Y \\ z & R_1 & R_2 & I \end{array}$$

$$\text{(ii)} \quad \begin{array}{c|ccc} & x' & y' & z' \\ \hline x & D & D & X \\ y & D & D & Y \\ z & C_1 & C_2 & I \end{array}$$

$$\text{(iii)} \quad \begin{array}{c|ccc} & x' & y' & z' \\ \hline x & D & D & R_1 \\ y & D & D & R_2 \\ z & X & Y & I \end{array}$$

$$\text{(iv)} \quad \begin{array}{c|ccc} & x' & y' & z' \\ \hline x & D & D & C_1 \\ y & D & D & C_2 \\ z & X & Y & I \end{array}$$

where X and Y are in two distinct fragments of $H(D)$ and I is such that no D lies below or to the right of it. Then in each case F has a configuration of the form

$$\begin{array}{|ccc} \hline D & D & R_i \\ D & D & C_j \\ C_k & R_m & I \end{array}\ .$$

Proof. We shall prove that in Cases (i) and (ii) F has a configuration of the form

$$\left|\begin{matrix} D & D & X \\ D & D & Y \\ C_k & R_m & I \end{matrix}\right. .$$

By taking the converse of the digraph D and by interchanging the labellings of R_i and C_i it will follow that in Cases (iii) and (iv) F has a configuration

$$\left|\begin{matrix} D & D & R_k \\ D & D & C_m \\ X' & Y' & I \end{matrix}\right.$$

where X' and Y' are the conjugate of X and Y respectively. Then the existence of any of the four forms in F will imply the existence of the form

$$\left|\begin{matrix} D & D & R_i \\ D & D & C_j \\ C_k & R_m & I \end{matrix}\right.$$

and the lemma will be proved.

Case (i): *The matrix F has a submtrix*

$$\begin{array}{c|cc} & w' & x' \\ \hline z & D & R_1 \\ w & C_1 & D \end{array} .$$

This C_1 must be below and to the left of the given R_1, as noted in the introduction for a digraph of F.D.2. So F has a configuration

$$\begin{array}{c|cccc} & w' & x' & y' & z' \\ \hline x & \cdot\cdot & D & D & X \\ y & \cdot\cdot & D & D & Y \\ z & D & R_1 & R_2 & I \\ w & C_1 & D & \cdot\cdot & \cdot\cdot \end{array}$$

where '..' means we are yet to be definite whether this entry is D or $\bar{D}$. Since H_1 and H_2 are two distinct components of $H_b(D)$, so xw' and yw' positions are both D and wy' position is $\bar{D}$. Again since X and Y are in two distinct fragments of $H(D)$, so wz' is

$\bar{D}$. So the above configuration takes the form

$$\begin{array}{c|cccc} & w' & x' & y' & z \\ \hline x & D & D & D & X \\ y & D & D & D & Y \\ z & D & R_1 & R_2 & I \\ w & C_1 & D & \bar{D} & \bar{D}. \end{array}$$

Again F has a submatrix

$$\begin{array}{c|ccc} & v' & x' & y' \\ \hline z & D & R_1 & R_2 \\ v & C_2 & \cdot\cdot & D\ . \end{array}$$

Since H_1 and H_2 are two distinct components, so vx' is $\bar{D}$. Now two subcases arise regarding the positions of v-row and w-row:

(a) when w-row lies above v-row, and

(b) when v-row lies above w-row.

Below we consider only the Subcase (a); the other Subcase (b) can be similarly proved and hence is omitted.

Subcase (a): *w-row lies above v-row.*

In this case F has a configuration

$$\begin{array}{c|ccccc} & v' & w' & x' & y' & z' \\ \hline x & \cdot\cdot & D & D & D & X \\ y & \cdot\cdot & D & D & D & Y \\ z & D & D & R_1 & R_2 & I \\ w & .. & C_1 & D & \bar{D} & \bar{D} \\ v & C_2 & \cdot\cdot & \bar{D} & D & \bar{D}\ . \end{array}$$

That vz' is $\bar{D}$ follows from the fact that X and Y are in distinct fragments and vx' is $\bar{D}$. Since $wy' * vx'$, they are not I's and since there is a D below wy', it must be an R_m. Thus a configuration of F is

$$\begin{array}{c|ccc} & w' & y' & z' \\ \hline x & D & D & X \\ y & D & D & Y \\ w & C_1 & R_m & \bar{D}\ . \end{array}$$

The Case (i) will be proved if now we can show that wz' is an I. If not, then there must be an entry pp' such that $wz' * pp'$, so that two possibilities arising out of the obstruction are

$$\begin{array}{c|cc} & p' & z' \\ \hline w & D & \bar{D} \\ p & \bar{D} & D \end{array} \qquad \begin{array}{c|cc} & z' & p' \\ \hline p & D & \bar{D} \\ w & \bar{D} & D \end{array}.$$

We first consider the first possibility to arrive at a contradiction; for the first possibility a configuration is

$$\begin{array}{c|cccccc} & v' & w' & x' & p' & y' & z' \\ \hline x & \cdot\cdot & D & D & \cdot\cdot & D & X \\ y & \cdot\cdot & D & D & \cdot\cdot & D & Y \\ z & D & D & R_1 & \cdot\cdot & R_2 & I \\ w & \cdot\cdot & C_1 & D & D & R_m & \bar{D} \\ p & \cdot\cdot & \cdot\cdot & \cdot\cdot & \bar{D} & \cdot\cdot & D \\ v & C_2 & \cdot\cdot & \bar{D} & \cdot\cdot & D & \bar{D} \end{array}.$$

The p'-column in the above configuration has been taken preceding to y'-column, because wy' is R_m and no D can lie to the right of a R. Since ww' is C_1 and xw', yw', zw' are all D's, w-row cannot occur preceding to any of x, y and z-rows. So none of the positions xp' and yp' is $\bar{D}$, because of the F_2-property of the configuration. Again, because X and Y are in two distinct fragments of $H_b(D)$, none of these position can be D. So wz' is an I. For the other possibility

$$\begin{array}{c|cc} & z' & p' \\ \hline p & D & \bar{D} \\ w & \bar{D} & D \end{array}$$

a possible configuration is

$$\begin{array}{c|cccccc} & v' & w' & x' & z' & p' & y' \\ \hline p & \cdot\cdot & \cdot\cdot & \cdot\cdot & D & \bar{D} & \cdot\cdot \\ x & \cdot\cdot & D & D & X & \cdot\cdot & D \\ y & \cdot\cdot & D & D & Y & \cdot\cdot & D \\ z & D & D & R_1 & I & \cdot\cdot & R_2 \\ w & \cdot\cdot & C_1 & D & \bar{D} & D & R_m \\ v & C_2 & \cdot\cdot & \bar{D} & \bar{D} & \cdot\cdot & D \end{array}.$$

The p'-column and so z'-column has been taken preceding to y'-column because wy' is R_m and no D can lie to the right of it. Again the p-row has been taken above x and

y rows because otherwise

$$\begin{array}{c|cc} & z' & p' \\ \hline x & X & D \\ y & Y & D \\ p & D & \cdot\cdot \end{array}$$

violates F_2-property. There is no particular fixed ordering between x'-column and z'-column. Now, neither xp' nor yp' is $\bar{D}$ because of F_2-property and also since X and Y are distinct, none of them is D. So wz' is an I and Case (a) is proved.

Case (ii): *The matrix F has a submatrix*

$$\begin{array}{c|cc} & y' & w' \\ \hline w & D & R_2 \\ z & C_2 & D \end{array}.$$

This R_2 must lie above and to the right of the given C_2, as observed in the introduction. So F has a configuration

$$\begin{array}{c|cccc} & x' & y' & w' & z' \\ \hline x & D & D & \cdot\cdot & X \\ y & D & D & \cdot\cdot & Y \\ w & \cdot\cdot & D & R_2 & \cdot\cdot \\ z & C_1 & C_2 & D & I \end{array}.$$

Since H_1 and H_2 are two distinct components of $H(D)$, wx' position is D and the positions xw' and yw' are D. Again since X and Y are in two distinct fragments of $H(D)$, wz' is D. So F takes the form

$$\begin{array}{c|cccc} & x' & y' & w' & z' \\ \hline x & D & D & D & X \\ y & D & D & D & Y \\ w & \bar{D} & D & R_2 & \bar{D} \\ z & C_1 & C_2 & D & I \end{array}.$$

Again, F has a configuration

$$\begin{array}{c|ccc} & x' & y' & v' \\ \hline v & D & \cdot\cdot & R_1 \\ z & C_1 & C_2 & D \end{array}.$$

This R_1 must lie above and to the right of the given C_1. Since H_1 and H_2 are distinct, vy' is $\bar{D}$. Now two cases arise regarding the positions of v-row and w-row.

Subcase (a): *v-row lies above w-row.*
In this case, F has a configuration

$$\begin{array}{c|ccccc} & x' & y' & v' & w' & z' \\ \hline x & D & \bar{D} & \cdot\cdot & D & X \\ y & D & D & \cdot\cdot & D & Y \\ v & D & \bar{D} & R_1 & \cdot\cdot & \cdot\cdot \\ w & \bar{D} & D & \cdot\cdot & R_2 & \bar{D} \\ z & C_1 & C_2 & D & D & I \end{array} .$$

The w-row has been taken below x and y-rows, because otherwise from

$$\begin{array}{c|cc} & x' & y' \\ \hline w & \bar{D} & D \\ x \text{ or } y & D & D \end{array}$$

it follows that the entry in wx'-position violates F_2-property. Now $vy' * wx'$. Since there is a D to the right of wx'-position, so this must be a C_m. Thus a configuration of F is

$$\begin{array}{c|ccc} & x' & w' & z' \\ \hline x & D & D & D \\ y & D & D & Y \\ w & C_m & R_2 & \bar{D} \end{array} .$$

The Case (ii) will be proved if now we can show that wz' is an I. If not, then there must be an entry pp' such that $wz' * pp'$, so that two possibilitis arising out of this obstruction are

$$\begin{array}{c|cc} & p' & z' \\ \hline w & D & \bar{D} \\ p & \bar{D} & D \end{array} \quad \text{and} \quad \begin{array}{c|cc} & z' & p' \\ \hline p & D & \bar{D} \\ w & \bar{D} & D \end{array} .$$

We first consider the first possibility to arrive at a contradiction; for the first possibility a configuration is

$$\begin{array}{c|cccccc} & x' & y' & v' & p' & w' & z' \\ \hline x & D & D & \cdot\cdot & \cdot\cdot & D & X \\ y & D & D & \cdot\cdot & \cdot\cdot & D & Y \\ v & D & \bar{D} & R_1 & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ w & C_m & D & \cdot\cdot & D & R_2 & \bar{D} \\ p & \cdot\cdot & \cdot\cdot & \cdot\cdot & \bar{D} & \cdot\cdot & D \\ z & C_1 & C_2 & D & \cdot\cdot & D & I \end{array} .$$

The p'-column in the above configuration has been taken preceding to w'-column, for otherwise

$$\begin{array}{c|cc} & w' & p' \\ \hline w & R_2 & D \\ z & D & \cdot\cdot \end{array}$$

violates F_2-property. Since wx' is C_m and xx', yx' are D, w-row cannot occur preceding to any of x and y-rows. So none of the positions xp' and yp' can be $\bar{D}$ because of the F_2-property of the configuration. Again, because X and Y are in two distinct fragments of $H_b(D)$, none of the positions can be D. So wz' is an I. For the other possibility

$$\begin{array}{c|cc} & z' & p' \\ \hline p & D & \bar{D} \\ w & \bar{D} & D \end{array}$$

a possible configuration is

$$\begin{array}{c|cccccc} & x' & y' & v' & z' & p' & w' \\ \hline p & \cdot\cdot & \cdot\cdot & \cdot\cdot & D & \bar{D} & \cdot\cdot \\ x & D & D & \cdot\cdot & X & \cdot\cdot & D \\ y & D & D & \cdot\cdot & Y & \cdot\cdot & D \\ v & D & \bar{D} & R_1 & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ w & C_m & D & \cdot\cdot & \bar{D} & D & R_2 \\ z & C_1 & C_2 & D & I & \cdot\cdot & D \end{array}.$$

Since ww' is R_2, p'-column and so z'-column has been taken preceding to w'-column. Again p-row has been taken above x and y-rows because otherwise

$$\begin{array}{c|cc} & z' & w' \\ \hline x & X & D \\ y & Y & D \\ p & D & \cdot\cdot \end{array}.$$

violates F_2-property. Now, neither of xp' and yp' can be D because of the F_2-property and also since X and Y are distinct, none of them is D. So wz' is an I and Case (a) is proved. □

Proof of Proposition 4. Let, for a certain statisfactory bicolouration of $H_b(D)$, $I_r \cap I_c \neq \emptyset$. Then it follows from Proposition 3 that F has a configuration

$$\begin{pmatrix} D & D & R_i \\ D & D & C_j \\ C_k & R_m & I \end{pmatrix}$$

for the given bicolouration. It may so happen that by a different satisfactory bicolouration, the above I fails to become an interior edge to both the realized Ferrers digraphs. This is possible only when the given configuaration takes any of the four forms in the Lemma 1 by the new bicolouration. (Note that R_i, C_j belonging to the same column are in two distinct fragments and similarly R_m, C_k belonging to the same row are in two distinct framgments, and consequently X and Y again in the same row or column in the Lemma 1 obtained by a change of labellings of the colours must be in distinct fragments). But in those cases, the lemma shows that there exists a configuration of the form

$$\begin{pmatrix} D & D & R_i \\ D & D & C_j \\ C_k & R_m & I \end{pmatrix}$$

with reference to the new bicolouration and accordingly $I_r \cap I_c \neq \emptyset$ also for the new bicolouration. $\square$

Theorem 2. *Let D be a digraph of F.D.2 with a satisfactory bicolouration $(\boldsymbol{R}, \boldsymbol{C})$ of $H_b(D)$ and let H_1 and H_2 be subdigraphs of $\bar{D}$ given by $H_1 = \boldsymbol{R} \cup \boldsymbol{I}_r$ and $H_2 = \boldsymbol{C} \cup \boldsymbol{I}_c$. Then both H_1 and H_2 are Ferrers digraphs.*

Proof. We will show that $xx' \in H_i$, $yy' \in H_i \Rightarrow xy' \in H_i$ or $yx' \in H_i$ $(i = 1, 2)$. We shall prove it for the digraph H_1 and the other case will similarly follow. For the two edges xx' and yy' belonging to H_1 there are three possible alternatives: (i) both of them are R, (ii) one is R and other is I_r, (iii) both of them are I_r. We consider the three cases below separately.

Case (i): *Let*

$$\begin{array}{c|cc} & x' & y' \\ \hline x & R & \cdot\cdot \\ y & \cdot\cdot & R \end{array}$$

be a configuration of the F_2-matrix F of A.

Two subcases arise: either the positions xy' and yx' are both $\bar{D}$ or one only of them is D, because otherwise xx' has an obstruction with yy' which is not possible since they have same colour.

(a) Let yx' be D; then xy' is not D. So

$$\begin{array}{c|cc} & x' & y' \\ \hline x & R & \bar{D} \\ y & D & R \end{array}$$

is a configuration of F. If xy' is an I, then it follows that it is an I_r. If xy' is not an I, then it follows that xy' is R (Proposition 1).

(b) Now assume that both xy' and yx' are $\bar{D}$. i.e.

$$\begin{array}{c|cc} & x' & y' \\ \hline x & R & \bar{D} \\ y & \bar{D} & R \end{array}.$$

Since $R \cup I(H)$ is a Ferrers digraph, both the $\bar{D}$'s cannot be C's. If any of them, say yx' is C or I, then the other must be an R because otherwise the configuration in $A(G_2)$

$$\begin{array}{|cc} x' & y' \\ \hline 0 & C/I \\ C/I & 0 \end{array}$$

gives an obstruction $yx' * xy'$. Hence either xy' or yx' is R (Proposition 1).

Case (ii): *Let*

$$\begin{array}{c|cc} & x' & y' \\ \hline x & R & \cdot\cdot \\ y & \cdot\cdot & I_r \end{array}$$

be a configuration of the F_2-matrix F of A.

As earlier, two subcases arise: (a) Let yx' (or xy') be D; then xy' (or yx') is $\bar{D}$, and so the two possible configurations are

$$\begin{array}{c|cc} & x' & y' \\ \hline x & R & \bar{D} \\ y & D & I_r \end{array} \quad \text{and} \quad \begin{array}{c|cc} & y' & x' \\ \hline x & D & R \\ y & I_r & \bar{D} \end{array}.$$

We show below that when yx' is D then xy' is either R or I_r. Since I_r is an interior edge, A must have a configuration

$$\begin{array}{c|cc} & z' & y' \\ \hline y & R & I_r \\ x & D & R \end{array}.$$

So

$$\begin{array}{c|ccc} & x' & z' & y' \\ \hline x & R & \cdot\cdot & \bar{D} \\ y & D & R & I_r \\ z & \cdot\cdot & D & R \end{array}$$

is a configuration of F. The $\bar{D}$ in the position xy' is an I or R or C. If xy' is a C then it can be seen that any possible permutation of the two rows and columns in the above configuration violates the condition of an F_2-matrix. Next when xy' is an I, the configuration

$$\begin{array}{c|cc} & x' & z' \\ \hline y & R & I \\ z & \cdot\cdot & R \end{array}$$

shows that zx' is either D or R (Proposition 1). Now from the configuration

$$\begin{array}{c|cc} & x' & z' \\ \hline y & D & R \\ z & \cdot\cdot & D, \end{array}$$

it follows that zx' is not R. So it must be D. Hence

$$\begin{array}{c|cc} & x' & y' \\ \hline x & R & I \\ z & D & R \end{array}$$

shows that xy' is an I in G. The $\bar{D}$ in the other possible configuration

$$\begin{array}{c|cc} & y' & x' \\ \hline x & D & R \\ y & I_r & \bar{D} \end{array}$$

can be similarly seen to be either R or I_r. Thus in any case the $\bar{D}$ in the configuration is an R or I_r and the theorem is proved for the Case (a).

(b) Now assume that both xy' and yx' are $\bar{D}$ so that

$$\begin{array}{c|cc} & x' & y' \\ \hline x & R & \bar{D} \\ y & \bar{D} & I_r \end{array}$$

is a configuration of F. Both of them cannot be C (Proposition 1). If one is R then nothing is to be proved. Now, let one of them, say xy', be I. The case when yx' is I can smilarly be taken care of. For the interior edge I_r on yy'-position, F must have a configuration

$$\begin{array}{c|cc} & z' & y' \\ \hline y & R & I_r \\ z & \bar{D} & R. \end{array}$$

So we have

$$\begin{array}{c|ccc} & x' & z' & y' \\ \hline x & R & \cdot\cdot & I \\ y & \bar{D} & R & I_r \\ z & \cdot\cdot & D & R. \end{array}$$

From the configuration

$$\begin{array}{c|cc} & x' & y' \\ \hline x & R & I \\ z & \cdot\cdot & R \end{array}$$

it follows that zx' must be either D or R (Proposition 1). If zx' is D, then it immediately follows that xy' is an interior edge I_r. If zx' is R, then the configuration

$$\begin{array}{c|cc} & x' & z' \\ \hline y & \bar{D} & R \\ z & R & D \end{array}$$

shows that yx' is either R or I [Case (i)].

Case (iii). *Let*

$$\begin{array}{c|cc} & x' & y' \\ \hline x & I_r & \cdot\cdot \\ y & \cdot\cdot & I_r \end{array}$$

be a configuration of F.

As earlier, the positions xy' and yx' are both $\bar{D}$ or only one of them is D.

(a) if yx' is D then xy' is $\bar{D}$ and

$$\begin{array}{c|cc} & x' & y' \\ \hline x & I_r & \bar{D} \\ y & D & I_r \end{array}$$

is a configration of F. For the interior edges I_r on xx' and yy' positions, there must exist configurations of the form

$$\begin{array}{c|cc} & w' & x' \\ \hline w & D & R \\ x & R & I_r \end{array} \qquad \text{and} \qquad \begin{array}{c|cc} & z' & y' \\ \hline z & D & R \\ y & R & I_r \end{array}$$

So F has a configuration

$$\begin{array}{c|cccc} & z' & w' & x' & y' \\ \hline z & D & \cdot\cdot & \cdot\cdot & R \\ w & \cdot\cdot & D & R & \cdot\cdot \\ x & \cdot\cdot & R & I_r & \bar{D} \\ y & R & \cdot\cdot & D & I_r \end{array}.$$

The z-row and the w-row cannot coincide, because in that case the configuration

$$\begin{array}{c|cc} & z' & x' \\ \hline z=w & D & R \\ y & R & D \end{array}$$

shows that $wx' * yz'$, which is not possible since they have the same colour. Similarly, z'-column and w'-column are distinct. From the configuration

$$\begin{array}{c|cc} & x' & y' \\ \hline w & R & \cdot\cdot \\ y & D & I_r \end{array}$$

it follows [by Case (ii)] that wy' is either R or I_r. So F takes the form

$$\begin{array}{c|cccc} & z' & w' & x' & y' \\ \hline z & D & \cdot\cdot & \cdot\cdot & R \\ w & \cdot\cdot & D & R & R/I_r \\ x & \cdot\cdot & R & I_r & \bar{D} \\ y & R & \cdot\cdot & D & I_r \end{array}$$

Now the configuration

$$\begin{array}{c|cc} & w' & y' \\ \hline w & D & R/I_r \\ x & R & \bar{D} \end{array}$$

shows that xy' is either R or I_r [by Cases (i) and (ii)]. The other possibility when xy' is D can be taken care of similarly.

(b) Next, let both xy' and yx' be $\bar{D}$; then F has a configuration

$$\begin{array}{c|cc} & x' & y' \\ \hline x & I_r & \bar{D} \\ y & \bar{D} & I_r\,. \end{array}$$

By Proposition 1, both xy' and yx' cannot be C. So one must be I or R. If it is R, then the theorem is proved. Let, one of xy' and yx', say, xy', be I. For the interior edges I_r on xx' and yy' positions, there must exist configurations of the form

$$\begin{array}{c|cc} & w' & x' \\ \hline w & D & R \\ x & R & I_r \end{array} \quad \text{and} \quad \begin{array}{c|cc} & z' & y' \\ \hline z & D & R \\ y & R & I_r \end{array}$$

in F. As earlier, z-row (z'-column) and w-row (w'-column) are distinct. So F has a configuration

$$\begin{array}{c|cccc} & z' & w' & x' & y' \\ \hline z & D & \cdot\cdot & \cdot\cdot & R \\ w & \cdot\cdot & D & R & \cdot\cdot \\ x & \cdot\cdot & R & I_r & I \\ y & R & \cdot\cdot & \bar{D} & I_r \end{array}.$$

For the configuration

$$\begin{array}{c|cc} & w' & y' \\ \hline x & R & I \\ y & \cdot\cdot & I_r \end{array}$$

if yw' is D then it follows that xy' is I_r [by Case (iia)] and the theorem is proved; if yw' is $\bar{D}$ then by Case (iib), yw' is either R or I_r. So F takes the form

$$\begin{array}{c|cccc} & z' & w' & x' & y' \\ \hline z & D & \cdot\cdot & \cdot\cdot & R \\ w & \cdot\cdot & D & R & \cdot\cdot \\ x & \cdot\cdot & R & I_r & I \\ y & R & R/I_r & \bar{D} & I_r \end{array}.$$

From the submatrix

$$\begin{array}{c|cc} & w' & x' \\ \hline w & D & R \\ y & R/I_r & \bar{D} \end{array}$$

it follows that yx' is either R or I_r [Cases (i) and (ii)].

Theorem 3. *Let D be a digraph of F.D.2. If D is an interval digraph, then for any satisfactory bicolouration of $H_b(D)$,*

$$I_r \cap I_c = \emptyset.$$

Proof. Let, if possible, $I_r \cap I_c \neq \emptyset$ for some satisfactory bicolouration of $H_b(D)$. Then there exist an $I \in \boldsymbol{I}(H)$ such that $I \in \boldsymbol{I}_r \cap \boldsymbol{I}_c$. It follows that I is an interior edge of both the Ferrers digraphs G_1 and G_2 where $G_1 = \boldsymbol{R} \cup \boldsymbol{I}(H)$ and $G_2 = \boldsymbol{C} \cup \boldsymbol{I}(H)$. First consider the case when $H_b(D)$ consists of one component only. Since the bicolouration of $H_b(D)$ is unique, R's and C's must belong exclusively to G_1 and G_2 respectively for any realization of $\bar{D}$ as the union of two Ferrers digraphs G_1 and G_2; it follows that the concerned I cannot be excluded from either of G_1 and G_2 for any such realization of $\bar{D}$. Hence $\bar{D}$ cannot be expressed as the union of two disjoint Ferrers digraphs and accordingly D cannot be an interval digraph.

Next, when $H_b(D)$ has more than one component, if $\boldsymbol{I}_r \cap \boldsymbol{I}_c \neq \emptyset$ for some satisfactory bicolouration, then the digraph $\bar{D}$ cannot be decomposed into two (disjoint) Ferrers digraphs with respect to the given bicolouration. Then the theorem follows from the Proposition 4 that $\boldsymbol{I}_r \cap \boldsymbol{I}_c \neq \emptyset$ for a bicolouration implies $\boldsymbol{I}_r \cap \boldsymbol{I}_c \neq \emptyset$ for any satisfactory bicolouration. □

That the converse of the above theorem is not true follows from the following counter-example.

Example 1. Consider the digraph $D(V, E)$ whose adjacency matrix is given by Fig. 2.

This is a digraph of F.D.2 and $H_b(D)$ has only one component for this digraph. By labelling the $\bar{D}$'s in terms of R's, C's and I's, F takes the form given in Fig. 3.

$$
\begin{array}{c|cccccccc}
 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \\
\hline
v_1 & D & D & D & D & D & \bar{D} & \bar{D} & \bar{D} \\
v_2 & D & D & D & D & \bar{D} & D & \bar{D} & \bar{D} \\
v_3 & D & D & D & D & \bar{D} & D & D & \bar{D} \\
v_4 & D & \bar{D} & \bar{D} & \bar{D} & \bar{D} & \bar{D} & \bar{D} & \bar{D} \\
v_5 & \bar{D} & D & \bar{D} & \bar{D} & \bar{D} & \bar{D} & \bar{D} & \bar{D} \\
v_6 & \bar{D} & D & D & D & \bar{D} & D & \bar{D} & D \\
v_7 & \bar{D} & \bar{D} & D & D & \bar{D} & \bar{D} & \bar{D} & \bar{D} \\
v_8 & \bar{D} & \bar{D} & \bar{D} & D & \bar{D} & D & \bar{D} & \bar{D}
\end{array}
$$

Fig. 2

$$
\begin{array}{c|cccccccc}
 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \\
\hline
v_1 & D & D & D & D & D & R & R & R \\
v_2 & D & D & D & D & C & D & I & R \\
v_3 & D & D & D & D & C & D & D & R \\
v_4 & D & R & R & R & I & R & I & R \\
v_5 & C & D & R & R & I & R & I & I \\
v_6 & C & D & D & D & C & D & C & D \\
v_7 & C & C & D & D & I & R & I & I \\
v_8 & C & C & C & D & C & D & I & I\,.
\end{array}
$$

Fig. 3

	v_1	v_5	v_2	v_3	v_4	v_7	v_6	v_8
v_4		I	R	R	R	I_r	R	R
v_5				R	R	I_r	R	I_r
v_1						R	R	R
v_7							R	I_r
v_2								R
v_3						I		R
v_6								
v_8								

H_1

	v_5	v_1	v_7	v_2	v_3	v_4	v_6	v_8
v_1								
v_4	I							
v_2	C		I					
v_3	C							
v_5	I_c	C						
v_6	C	C	C					
v_7	I_c	C	I_c	C				
v_8	C	C	I_c	C	C			

H_2

Fig. 4.

The set of interior edges of $G_1 = R \cup I(H)$ and $G_2 = C \cup I(H)$ are $I_r(G_1) = \{v_4v_7, v_5v_7, v_5v_8, v_7v_8\}$ and $I_c(G_2) = \{v_5v_5, v_7v_5, v_7v_7, v_8v_7\}$. It is seen that $I_r \cap I_c = \emptyset$ for this digraph. If $H_1 = R \cup I_r$ and $H_2 = C \cup I_c$, then $H_1 \cup H_2 \neq \bar{D}$ and the two edges v_4v_5 and v_2v_7 lie outside $H_1 \cup H_2$.

The representation of H_1 and H_2 along with the two edges v_4v_5 and v_2v_7 in the form of a matrix (see Fig. 4) makes it clear.

Since the bicolouration is unique, H_1 and H_2 must be contained exclusively in any two decomposed Ferrers digraphs of $\bar{D}$. Nevertheless, the edges v_2v_7 cannot be adjoined to any of H_1 and H_2 to make them Ferrers digraphs again. Hence it is not possible to cover $\bar{D}$ by two disjoint Ferrers digraphs and accordingly D is not an interval digraph.

Conclusion

We conclude the paper leaving unresolved the problem of characterizing an interval digraph in terms of the notion of interior edges with reference to a decomposition of the digraph into two Ferrers digraphs.

Acknowledgement

The authors are grateful to the referees for their careful reading of the paper and for excellent suggestions that has led to an overall reorganization and a qualitative improvement of the paper as a whole.

References

[1] L.W. Beineke and C.M. Zamfirescu, Connection digraphs and second-order line graphs, Discrete Math. 39 (1982) 237–254.

[2] A. Bouchet, Etude combinatoire des ordonnes finis applications. Theses d'Etat, Grenoble, 1971.
[3] O. Cogis, Ferrers digraphs and threshold graphs, Discrete Math. 38 (1982) 33–46.
[4] O. Cogis, On the Ferrers dimension of a digraph, Discrete Math. 38 (1982) 47–52.
[5] O. Cogis, A characterization of digraphs with Ferrers dimension 2, Rapport de Recherche, no 19, GR. CNRS no. 22 Paris, 1979.
[6] J.F. Doignon, A. Ducamp and J.C. Falmagne, On realizable biorders and the biorder dimension of a realation, J. Math. Phych. 28 (1984) 73–109.
[7] A. Ducamp and J.C. Falmagne, Composite measurement, J. Math. Phych. 6 (1969) 359–390.
[8] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs (Acadmic Press, New York, 1980).
[9] L. Guttman, A basis for scaling quantitative data, Amer Soc. Rev. 9 (1944) 139–150.
[10] J. Riguet, Les relations de Ferrers, C.R. Acad, Sci. Paris 232 (1951) 1729–1730.
[11] M. Sen, S. Das, A.B. Roy and D.B. West, Interval digraphs: An analogue of interval graphs, J. Graph Theory 13 (1989) 189–202.
[12] M. Sen, S. Das and D.B. West, Circular-arc digraphs: A characterization, J. Graph Theory 13 (1989) 581–592.
[13] D.B. West, Parameters of partial orders and graphs: packing, covering and representation, in: I. Rival, ed., Graphs and Order (Reidel, Boston, 1985) 267–350.